Problem 1) After Ed eats 20% of his pie and Ann eats 40% of her pie, Ed has twice as much pie left as Ann. Find Ed's original amount of pie as a percentage of Ann's original pie.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: 2011 AMATYC Test #2]

Solution: Let x be the original amount of Ed's pie and y be the original amount of Ann's pie. Then the first sentence of the problem may be written as this equation: $.8x = 2(.6y)$.

So, $y = \frac{2}{3}x$. Ed's original amount of pie as a percentage of Ann's original pie is

$\frac{x}{y} \times 100\%$ and $\frac{x}{y} \times 100\% = \frac{x}{\frac{2}{3}x} \times 100\% = 150\%$.

Problem 2) Express $\dfrac{\sqrt{4+2\sqrt{3}}-\sqrt{28+10\sqrt{3}}}{15}$ as a rational number.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary's College Mathematics Contest Problems for Junior and Senior High School by Brother Alfred Brousseau, 1972]

Solution: Note that $4+2\sqrt{3}=(1+\sqrt{3})^2$ and $28+10\sqrt{3}=(5+\sqrt{3})^2$. So,

$$\frac{\sqrt{4+2\sqrt{3}}-\sqrt{28+10\sqrt{3}}}{15}=\frac{1+\sqrt{3}-(5+\sqrt{3})}{15}=\frac{-4}{15}$$

Problem 3) If b varies over all real numbers, upon what curve do the vertices of the parabolas with equations $f(x) = x^2 + bx + 2$ lie?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: 1989 AMATYC Exam #1]

Solution: The x-coordinate of the vertex of $f(x) = x^2 + bx + 2$ is i) $x = -\frac{b}{2}$. The y-coordinate of the vertex is $y = f(-\frac{b}{2})$. So, ii) $y = 2 - \frac{b^2}{4}$. From i) $b = -2x$. Substitute into ii) to get $y = 2 - x^2$. The graph of this equation, a downward opening parabola whose vertex is (0,2), is the answer to the question.

Problem 4) The volumes of two cubes differ by 259 cm^3. If the edges of one cube are each 4 cm greater than the edges of the other, find sum of the lengths of one edge of each cube.

[Problem submitted by Anatoliy Nikolaychuk, LACC Professor of Mathematics. Source: February 2000 AMATYC]

Solution:

$$(x+4)^3 - x^3 = 259$$
$$x^2 + 16x - 65 = 0$$
$$(2x-5)(2x+13) = 0$$
$$x = \frac{5}{2},\ x+4 = \frac{13}{2}$$
$$x + (x+4) = \frac{5}{2} + \frac{13}{2} = 9$$

Problem 5) Sand is pouring from a funnel onto a cone-shaped pile which is 20 feet high and 50 feet in diameter at its base. If the sand is coming out of the funnel at a rate of 2.5 cubic feet per second, how long will it take for the height of the pile to increase 1 inch, assuming that the shape of the pile does not change?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary's College Mathematics Contest Problems for Junior and Senior High School by Brother Alfred Brousseau, 1972]

Solution: Let x be the change in the radius of the base. Then

$$\frac{20}{25} = \frac{20 + \frac{1}{12}}{25 + x}$$

$$x = \frac{5}{48}.$$

The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

The change in volume is $\Delta V = \frac{\pi}{3}(25 + \frac{5}{48})^2 (20 + \frac{1}{12}) - \frac{\pi}{3} \cdot 25^2 \cdot 20 \approx 164.3$.

$$time \approx \frac{164.3}{2.5} \approx 65.7 \text{ seconds or } time \approx 20.9\pi \text{ seconds}$$

Problem 6) Suppose x and $f(x)$ are real numbers. Find the inverse function of $f(x)=x+\sqrt{x}$. That is, find a function $f^{-1}(x)$ such that $f^{-1}[f(x)]=x$ for every x in the domain of $f(x)$. Prove your answer is correct.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: Let $y=f(x)$ and solve $y=x+\sqrt{x}$ for x:

$0=x+\sqrt{x}-y$. Use the quadratic formula to get

$\sqrt{x}=\dfrac{-1\pm\sqrt{1+4y}}{2}$. So,

$x=\left(\dfrac{-1\pm\sqrt{1+4y}}{2}\right)^2$. Since $f(x)$ is a one-to-one function, its inverse is a function. This gives two possibilities for $f^{-1}(x)$. Either

A) $f^{-1}(x)=\left(\dfrac{-1-\sqrt{1+4x}}{2}\right)^2$ or B) $f^{-1}(x)=\left(\dfrac{-1+\sqrt{1+4x}}{2}\right)^2$, but not both. Consider the first possibility:

$$\begin{aligned}
\text{A) } f^{-1}(x) &= \left(\frac{-1-\sqrt{1+4x}}{2}\right)^2 \\
&= \frac{1}{2}+\frac{1}{2}\sqrt{1+4x}+x. \text{ Then} \\
f^{-1}[f(x)] &= \frac{1}{2}+\frac{1}{2}\sqrt{1+4(x+\sqrt{x})}+x+\sqrt{x} \\
&= \frac{1}{2}+\frac{1}{2}\sqrt{(2\sqrt{x}+1)^2}+x+\sqrt{x} \\
&= x+2\sqrt{x}+1 \\
&\neq x.
\end{aligned}$$

So, A) is not the inverse function. Now consider the second possibility:

$$\begin{aligned}
\text{B) } f^{-1}(x) &= \left(\frac{-1+\sqrt{1+4x}}{2}\right)^2 \\
&= \frac{1}{2}-\frac{1}{2}\sqrt{1+4x}+x. \text{ Then} \\
f^{-1}[f(x)] &= \frac{1}{2}-\frac{1}{2}\sqrt{1+4(x+\sqrt{x})}+x+\sqrt{x} \\
&= \frac{1}{2}-\frac{1}{2}\sqrt{(2\sqrt{x}+1)^2}+x+\sqrt{x} \\
&= x.
\end{aligned}$$

Therefore, B) is the inverse function.

Problem 7) Find the sum: $\sum_{k=1}^{\infty} \frac{k}{8^k}$.

[Problem submitted by Anatoliy Nikolaychuk, LACC Professor of Mathematics. Source: February 2000 AMATYC]

Solution: Let $S = \sum_{k=1}^{\infty} \frac{k}{8^k}$. Then

$$S = \frac{1}{8} + \frac{2}{8^2} + \frac{3}{8^3} + \frac{4}{8^4} + \cdots$$

$$\frac{S}{8} = \frac{1}{8^2} + \frac{2}{8^3} + \frac{3}{8^4} + \cdots$$

$$S - \frac{S}{8} = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \frac{1}{8^4} + \cdots$$

$$\frac{7}{8} S = \sum_{k=1}^{\infty} \frac{1}{8^k}$$

$$\frac{7}{8} S = \frac{\frac{1}{8}}{1 - \frac{1}{8}}$$

$$S = \frac{8}{49}$$

Problem 8) Suppose $x \geq 0$. Find the inverse function of $f(x) = 2^{x-1} + 2^{-(x+1)}$. That is, find a function $f^{-1}(x)$ such that $f^{-1}[f(x)] = x$ for every x in the domain of $f(x)$. Also, find the domain and range of $f^{-1}(x)$.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: Note that the domain of $f(x)$ is $[0, \infty)$. $f(x)$ is increasing and $f(0) = 1$. [By plotting a few points of $f(x)$ it can be seen to be increasing or it can be proved to be increasing on $(1, \infty)$ by taking its derivative.] So, the range of $f(x)$ is $[1, \infty)$. Therefore, the domain of $f^{-1}(x)$ is $[1, \infty)$ and its range is $[0, \infty)$. Let $y = f(x)$ and solve $y = 2^{x-1} + 2^{-(x+1)}$ for x. First multiply both sides of the equation by 2^{x+1} to get

$$2y2^x = 2^{2x} + 2^0$$

$$0 = 2^{2x} - 2y2^x + 1$$. Use the quadratic formula to get

$$2^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$$

$$2^x = y \pm \sqrt{y^2 - 1}$$. Because $x \geq 0$, $2^x \geq 1$. However,

$y - \sqrt{y^2 - 1}$ is not greater than or equal to 1 for every $y \in [1, \infty)$. For example $2 - \sqrt{2^2 - 1} = 2 - \sqrt{3} \approx .27$. * Therefore,

$$2^x = y + \sqrt{y^2 - 1}$$

$$x = \log_2(y + \sqrt{y^2 - 1})$$. Substitute $f^{-1}(x)$ for x and x for y to get

$$f^{-1}(x) = \log_2(x + \sqrt{x^2 - 1}).$$

* Or prove that $y - \sqrt{y^2 - 1} < 1$ for every $y \in (1, \infty)$:

$$1 < y$$

$$1 - y < 0$$

$$2 - 2y < 0$$

$$1 - 2y < -1$$

$$1 - 2y + y^2 < -1 + y^2$$

$$(y-1)^2 < y^2 - 1$$

$$y - 1 < \sqrt{y^2 - 1}$$

$$y - \sqrt{y^2 - 1} < 1$$

Problem 9) In the equation $x^4 - 4x^3 - 12x^2 - 13x + 20 = 0$ what is the sum of the squares of the four roots (solutions)?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary's College Mathematics Contest Problems for Junior and Senior High School by Brother Alfred Brousseau, 1972]

Solution: Let a, b, c, and d be the roots of the equation. Then $(x-a)(x-b)(x-c)(x-d) = x^4 - 4x^3 - 12x^2 - 13x + 20$. Now multiply to get the following.

$$(x-a)(x-b) = x^2 - (a+b)x + ab$$
$$(x-c)(x-d) = x^2 - (c+d)x + cd$$

Next multiply $[x^2 - (a+b)x + ab][x^2 - (c+d)x + cd]$ to get

$$x^4 - (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 - [ab(c+d) + cd(a+b)]x + abcd$$

Equating the third and second degree coefficients in this equation with those in the equation given in the problem, we get $a+b+c+d = 4$ and $ab+ac+ad+bc+bd+cd = -12$. These two values will be substituted into an equation below.

Consider $(a+b+c+d)^2$. Expand to get this equation:

$$(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab+ac+ad+bc+bd+cd).$$

So, $4^2 = a^2 + b^2 + c^2 + d^2 + 2(-12)$ and $a^2 + b^2 + c^2 + d^2 = 40$.

Problem 10) Let $\Diamond$ be an operation (like addition or multiplication) which associates each pair x,y of real numbers with the real number x $\Diamond$ y such that for all real numbers x, y, z the following conditions are satisfied; 1) $x\Diamond x = x$, 2) $x\Diamond y = y\Diamond x$, 3) $x\Diamond(y\Diamond z) = (x\Diamond y)\Diamond z$, and 4) if $y < z$ and $x\Diamond y \neq x$ then $x\Diamond y < x\Diamond z$. Prove that for every pair of real numbers x,y $x\Diamond y = x$ or $x\Diamond y = y$. Also, find an operation that satisfies these four conditions.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: 1994-95 Pomona-Wisconsin Talent Search, Problem Set IV]

Solution: Assume that there exists a pair of real numbers x and y such that $x\Diamond y \neq x$ and $x\Diamond y \neq y$. From condition 1) we know $x \neq y$. Without loss of generality say $y < x$. Then from conditions 1) and 4) we can conclude that $x\Diamond y < x\Diamond x = x$; that is, A) $x\Diamond y < x$.

Now using conditions 1), 2), and 3) we conclude that $y\Diamond(x\Diamond y) = (y\Diamond x)\Diamond y = (x\Diamond y)\Diamond y = x\Diamond(y\Diamond y) = x\Diamond y$; that is, B) $y\Diamond(x\Diamond y) = x\Diamond y$.

Next using our assumption $x\Diamond y \neq y$, A), B), and condition 4) we get that $x\Diamond y < x$ and $y\Diamond(x\Diamond y) \neq y$ implies that $y\Diamond(x\Diamond y) < y\Diamond x$. Then according to B) and condition 1) $x\Diamond y < x\Diamond y$ which is a contradiction.

So, our original assumption that there exists a pair of real numbers x,y such that $x\Diamond y \neq x$ and $x\Diamond y \neq y$ must be false. Therefore, for every pair of real numbers x,y $x\Diamond y = x$ or $x\Diamond y = y$.

Two operations satisfying all four of the given conditions are max{x,y} and min{x,y}.