Problem 1 The sum of two numbers is 100. The larger number minus the smaller number is 27. Find the numbers.
[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: Let x be the larger number and y the smaller number.

$$x + y = 100$$
$$x - y = 27$$
$$\therefore 2x = 127 \rightarrow x = 63\frac{1}{2} \rightarrow y = 100 - 63\frac{1}{2} = 36\frac{1}{2}$$

Problem 2 The ratio of the legs of a right triangle is 3 to 4. If the hypotenuse of the right triangle is 30 cm, what are the measures of the legs?
[Problem submitted by Kee Lam, LACC Professor of Mathematics. Source: Kee Lam]

Solution: Let the measures of the legs of the right triangle are 3x and 4x, then by Pythagorean Theorem,

$$\begin{aligned}(3x)^2 + (4x)^2 &= (30)^2 \\ 9x^2 + 16x^2 &= 900 \\ 25x^2 &= 900 \\ x^2 &= 36 \\ x &= 6\end{aligned}$$

Hence, the measures of the legs of the right triangle are $3(6) = 18cm$ and $4(6) = 24cm$

Problem 3 Find the value of x if $\sqrt[3]{x+1}-\sqrt[3]{x-1}=1$.
[Problem submitted by Kee Lam, LACC Professor of Mathematics. Source: Kee Lam]

Solution: Cube both sides of the equation and then simplify:

$$(x+1)-3(x+1)^{2/3}(x-1)^{1/3}+3(x+1)^{1/3}(x-1)^{2/3}-(x-1)=1$$

$$2-3(x+1)^{2/3}(x-1)^{1/3}+3(x+1)^{1/3}(x-1)^{2/3}=1$$

$$-3(x+1)^{1/3}(x-1)^{1/3}\left[(x+1)^{1/3}-(x-1)^{1/3}\right]=-1$$

$$3\left(x^2-1\right)^{1/3}=1$$

$$x^2-1=\frac{1}{27}$$

$$x^2=\frac{28}{27}$$

$$x=\pm\frac{2\sqrt{21}}{9}$$

Problem 4 The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. Find b.
[Problem submitted by Ha Nguyen, LACC Adjunct Instructor of Mathematics. Source: Mathematical Association of America's American Mathematics Competition 2010 Problem 11]

Solution:

$$7^{x+7} = 8^x$$

$$7^x \cdot 7^7 = 8^x$$

$$\left(\frac{8}{7}\right)^x = 7^7$$

$$x = \log_{\frac{8}{7}} 7^7$$

So, $b = \frac{8}{7}$.

Problem 5 If $a+\frac{1}{a}=10$, Find $a^3+\frac{1}{a^3}$.
[Problem submitted by Munir Samplewala, LACC Professor of Computer Science and Information Technology. Source: Munir Samplewala]

Solution: $$\left(a+\frac{1}{a}\right)^3 = 10^3$$

$$a^3+3a^2\left(\frac{1}{a}\right)+3a\left(\frac{1}{a}\right)^2+\frac{1}{a^3}=1000$$

$$a^3+3a+\frac{3}{a}+\frac{1}{a^3}=1000$$

$$a^3+\frac{1}{a^3}+3\left(a+\frac{1}{a}\right)=1000$$

$$a^3+\frac{1}{a^3}+3(10)=1000$$

$$a^3+\frac{1}{a^3}=970$$

Problem 6 A sequence $\{a_n\}$ satisfies $a_n = a_{n-1} + a_{n-3}$ for all $n \geq 4$. If $a_1 = 3$ and $a_6 = 30$, find a_8.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: AMATYC Student Mathematics League, October/November 2014.]

Solution: Let $a_2 = x$ and $a_3 = y$.

Then
$$a_4 = y + 3$$
$$a_5 = x + y + 3$$
$$a_6 = x + 2y + 3$$
$$a_7 = x + 2y + 3 + y + 3$$
$$a_8 = x + 2y + 3 + y + 3 + x + y + 3 = 2(x + 2y + 3) + 3 = 2(30) + 3 = 63$$

Problem 7 For the function $f(x)$, $f(1)=4$. Also, $f(x)f(y)=f(x+y)+f(x-y)$ for all real numbers x and y. Find $f(5)$.
[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: AMATYC Faculty Mathematics League, Test I, November 14, 2014.]

Solution:

$$f(1)f(0)=f(1+0)+f(1-0)$$
$$4f(0)=4+4 \quad \rightarrow \quad f(0)=2$$

$$f(1)f(1)=f(1+1)+f(1-1)$$
$$4\cdot 4=f(2)+2 \quad \rightarrow \quad f(2)=14$$

$$f(2)f(1)=f(2+1)+f(2-1)$$
$$14\cdot 4=f(3)+4 \quad \rightarrow \quad f(3)=52$$

$$f(3)f(2)=f(3+2)+f(3-2)$$
$$52\cdot 14=f(5)+4 \quad \rightarrow \quad f(5)=724$$

Problem 8 A cubic equation has three roots which are perfect squares such that $a^2 + b^2 = c^2$, where a^2, b^2, and c^2 are the three roots. If the equation is $x^3 + px^2 + qx + r = 0$, find the relation that holds among p, q, and r.
[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary's College Mathematics Contest Problems by Brother Alfred Brousseau, Creative Publications, 1972.]

Solution: Since a^2, b^2, and c^2 are the three roots, $(x-a^2)(x-b^2)(x-c^2) = 0$.
Multiply, collect like terms, and equate to the given cubic equation to get

$$x^3 - (a^2 + b^2 + c^2)x^2 + (a^2b^2 + a^2c^2 + b^2c^2)x - a^2b^2c^2 = x^3 + px^2 + qx + r \ .$$

Equate the coefficients of like terms to get

$$a^2 + b^2 + c^2 = -p \qquad \rightarrow \qquad c^2 = -\frac{1}{2}p$$

$$a^2b^2 + a^2c^2 + b^2c^2 = q \qquad \rightarrow \qquad a^2b^2 + c^2(a^2 + b^2) = q$$

$$a^2b^2 + c^4 = q$$

$$a^2b^2 + \frac{1}{4}p^2 = q$$

$$-a^2b^2c^2 = r \qquad \rightarrow \qquad -a^2b^2\left(-\frac{1}{2}p\right) = r$$

$$a^2b^2 = \frac{2r}{p}$$

Therefore, $\frac{2r}{p} + \frac{1}{4}p^2 = q$. So, $8r + p^3 = 4pq$.

Problem 9 Find all prime numbers of the form 100…001, where the number of zeros between the first and last digits is even.
[Problem submitted by Iris Magee, LACC Professor of Mathematics. Source: Wisconsin Mathematics Science & Engineering Talent Search.]

Solution: Note that if *m* is odd and positive, then

$$a^m + b^m = (a+b)\left(a^{m-1} - a^{m-2}b + a^{m-3}b^2 - a^{m-4}b^3 + \cdots + b^{m-1}\right).$$

Let k be the number of zeros in 100…001. Then $100...001 = 10^{k+1} + 1^{k+1}$.

Since k is even, k+1 is odd. So, 100…001 is factorable as shown above with one of the factors being (10+1). Therefore, 11 is the only prime number of the form 100…001 in which the number of zeros between the first and last digits is even.

Problem 10 For the sequence 113, 118, 109, 74, 1, -122,…, a_n,… find the polynomial of least degree, f(n), such that $a_n = f(n)$ for n = 1, 2, 3, ….
[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: For any sequence $a_1, a_2, a_3, a_4,...$ define the first difference sequence to be $a_2 - a_1, a_3 - a_2, a_4 - a_3, a_5 - a_4,....$ Define the second difference sequence to be the first difference sequence of the first difference sequence and so on.

Consider the sequence defined by $a_n = an + b$: $a + b, 2a + b, 3a + b,....$ Its first difference sequence is $a, a, a,....$

Next consider the sequence defined by $a_n = an^2 + bn + c$:
$a + b + c, 4a + 2b + c, 9a + 3b + c,....$ Its first difference sequence is
$3a + b, 5a + b + c, 7a + b,....$ Its second difference sequence is $2a, 2a, 2a,....$

Now consider the sequence defined by $a_n = an^3 + bn^2 + cn + d$:
$a + b + c + d, 8a + 4b + 2c + d, 27a + 9b + 3c + d,....$ Its third difference sequence is
$6a, 6a, 6a,....$

Therefore, if k = 1, 2, or 3 for a sequence whose terms are defined by a kth degree polynomial, the kth difference sequence is a constant sequence whose terms are $k!a$ where a is the leading coefficient of the polynomial.

Applying this observation to the given sequence in this problem,
113, 118, 109, 74, 1, -122,…, the first, second and third difference sequences are:

first: 5, -9, -35, -73, -123,…
second: -14, -26, -38, -50,…
third: -12, -12, -12,….

This implies the sequence is defined by a third degree polynomial:
$f(n) = an^3 + bn^2 + cn + d$ with $3!a = -12$. So, a = -2 and $d + cn + bn^2 = f(n) + 2n^3$.
Substitute n = 1, n = 2, and n = 3 into this equation to get three equations with three unknowns:
$d + c + b = 115$
$d + 2c + 4b = 134$
$d + 3c + 9b = 163$
Solve this system to get d = 106, c = 4, and b = 5.

Therefore, $a_n = f(n) = -2n^3 + 5n^2 + 4n + 106$.