Problem 1) For what values of k does this system of equations have at least one solution and for each value of k what is (are) the solution(s) to the system?

1) $x + ky = 3$
2) $kx + 2y = 2$
3) $x + y = 1$

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: 2001 AMATYC Student Mathematics League]

Solution: First subtract equation 3) from equation 1) to get $(k-1)y = 2$ which implies $y = \frac{2}{k-1}$. Next subtract k times equation 1) from equation 2) to get $(2-k^2)y = 2-3k$ which implies $y = \frac{2-3k}{2-k^2}$. For a particular k the system will have at least one solution if and only if these two y values are equal; that is, if and only if $\frac{2}{k-1} = \frac{2-3k}{2-k^2}$. So,

$$2(2-k^2) = (k-1)(2-3k)$$
$$-2k^2 + 4 = -3k^2 + 5k - 2$$
$$k^2 - 5k + 6 = 0$$
$$k = 2,3$$

If k = 2, then the system is
$x + 2y = 3$
$2x + 2y = 2$
$x + y = 1$, and the solution is (-1, 2).

If k = 3, then the system is
$x + 3y = 3$
$3x + 2y = 2$
$x + y = 1$, and the solution is (0, 1) .

Problem 2) Find the sum. Write the answer in simplest form.

$$1+\frac{2}{10}+\frac{3}{10^2}+\frac{7}{10^3}+\frac{2}{10^4}+\frac{3}{10^5}+\frac{7}{10^6}+\frac{2}{10^7}+\frac{3}{10^8}+\frac{7}{10^9}+\cdots$$

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Calculus, 11th edition, by James Stewart.]

Solution 1:

$$1+\frac{2}{10}+\frac{3}{10^2}+\frac{7}{10^3}+\frac{2}{10^4}+\frac{3}{10^5}+\frac{7}{10^6}+\frac{2}{10^7}+\frac{3}{10^8}+\frac{7}{10^9}+\cdots$$

$$=1+\frac{2}{10}\left(\frac{1}{10^0}+\frac{1}{10^3}+\frac{1}{10^6}+\cdots\right)+\frac{3}{10^2}\left(\frac{1}{10^0}+\frac{1}{10^3}+\frac{1}{10^6}+\cdots\right)+\frac{7}{10^3}\left(\frac{1}{10^0}+\frac{1}{10^3}+\frac{1}{10^6}+\cdots\right)$$

$$=1+\left(\frac{2}{10}+\frac{3}{10^2}+\frac{7}{10^3}\right)\left(\frac{1}{10^0}+\frac{1}{10^3}+\frac{1}{10^6}+\cdots\right)$$

$$=1+\frac{200+30+7}{10^3}\left(\frac{1}{10^0}+\frac{1}{10^3}+\frac{1}{10^6}+\cdots\right)$$

$$=1+\frac{237}{10^3}\left(\frac{1}{10^0}+\frac{1}{10^3}+\frac{1}{10^6}+\cdots\right)$$

$$=1+\frac{237}{10^3}\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n$$

$$=1+\frac{237}{10^3}\cdot\frac{1}{1-\frac{1}{10^3}}$$

$$=1+\frac{237}{10^3}\cdot\frac{10^3}{999}$$

$$=\frac{412}{333}$$

Solution 2:

$$1+\frac{2}{10}+\frac{3}{10^2}+\frac{7}{10^3}+\frac{2}{10^4}+\frac{3}{10^5}+\frac{7}{10^6}+\frac{2}{10^7}+\frac{3}{10^8}+\frac{7}{10^9}+\cdots$$

$$=1+(\frac{2}{10}+\frac{3}{10^2}+\frac{7}{10^3})+(\frac{2}{10^4}+\frac{3}{10^5}+\frac{7}{10^6})+(\frac{2}{10^7}+\frac{3}{10^8}+\frac{7}{10^9})+\cdots$$

$$=1+\frac{237}{10^3}+\frac{237}{10^6}+\frac{237}{10^9}+\cdots=1+\frac{237}{10^3}(1+\frac{1}{10^3}+\frac{1}{10^6}+\cdots)$$

$$=1+\frac{237}{10^3}\cdot\frac{1}{1-\frac{1}{10^3}}=1+\frac{237}{1000}\cdot\frac{1000}{999}=\frac{412}{333}$$

Problem 3) For a function $f(x)$, let $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, and so on. For the function $f(x) = \sqrt{\frac{x^2+1}{x^2-1}}$ on the domain $(-\infty,-1)\cup(1,\infty)$ find $f^{2011}(x)$.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: 2010 AMATYC competition]

Solution: Note that f(x) is an even function. So, $f(|x|) = f(x)$

$$f^2(x) = \sqrt{\frac{\frac{x^2+1}{x^2-1}+1}{\frac{x^2+1}{x^2-1}-1}} = \sqrt{\frac{x^2+1+x^2-1}{x^2+1-(x^2-1)}} = \sqrt{\frac{2x^2}{2}} = \sqrt{x^2} = |x|$$

$$f^3(x) = f(|x|) = f(x)$$

So, for any positive odd exponent m, $f^m(x) = f(x)$. Therefore, $f^{2011}(x) = f(x) = \sqrt{\frac{x^2+1}{x^2-1}}$

Problem 4) Suppose an arborist is measuring the height of a tree, but cannot get to the base of the tree due to thick brush. At an unknown distance x from the base of the tree the arborist measures the angle to the top of the tree as 45 degrees above horizontal. Backing up 200 feet from where the first measurement was taken, the angle to the top of the tree is found to be 30 degrees. Assume the tree is vertical and the ground is horizontal from the tree to where the measurements were taken. Find the height of the tree. Write the answer in simplest form.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: Let h be the height of the tree. Then h and x form two equal sides of a 45 degree right triangle; that is, h = x. Also, h and h + 200 are two legs of a 30 degree right triangle. So,

$$\frac{h}{h+200} = \frac{1}{\sqrt{3}}$$

$$h = \frac{1}{\sqrt{3}}h + \frac{200}{\sqrt{3}}$$

$$h - \frac{1}{\sqrt{3}}h = \frac{200}{\sqrt{3}}$$

$$h = \frac{200}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}-1}$$

$$h = \frac{200}{\sqrt{3}-1}$$

$$h = \frac{200}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$h = \left(\sqrt{3}+1\right)100$$

Problem 5) Suppose a rider is pedaling a bicycle at an angular velocity of ω_1 measured in rpm (revolutions per minute), and the pedals are attached to a chain ring whose radius is r_1 measured in feet. The chain ring is connected by a bicycle chain to a back sprocket whose radius is r_2 feet which is turning at ω_2 rpm. This back sprocket is attached to the back wheel of the bicycle whose radius (including the tire) is R feet. Find V, the speed of the bike in feet per second in terms of ω_1, r_1, r_2, and R.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: The speed of the bicycle is the same as the linear velocity of a point on the outer edge of the rear tire, which in feet per minute is $V = 2\pi R\omega_2$. In feet per second the speed of the bicycle is

$$V = 2\pi R\omega_2 \cdot \frac{\text{min}}{60\,\text{sec}}$$

Since the chain ring and rear sprocket are connected by the chain, a point on the edge of either is moving at the same speed as the chain; that is, $\omega_2 r_2 = \omega_1 r_1$. So,

$$\omega_2 = \frac{\omega_1 r_1}{r_2}.$$

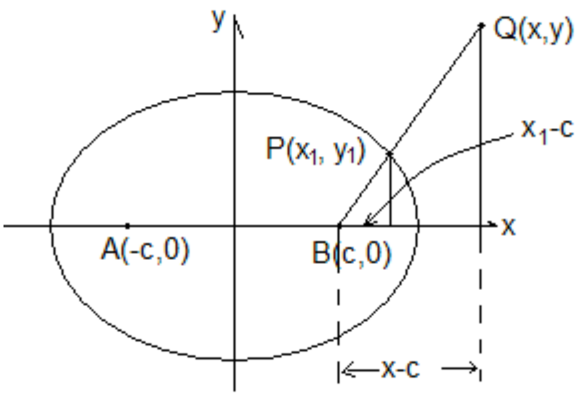
Therefore, the speed of the bike in feet per second in terms of ω_1, r_1, r_2, and R is

$$V = 2\pi R \cdot \frac{\omega_1 r_1}{r_2} \cdot \frac{\text{min}}{60\,\text{sec}} = \frac{\pi R\omega_1 r_1}{30 r_2}\ ft/\text{sec}$$

Problem 6) P is an arbitrary point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, A(-c,0) and B(c,0) are the foci of the ellipse. Draw a line segment from B to P and extend it to Q such that the distance from P to Q is twice the distance from B to P. Find the locus of Q.

[Problem submitted by Steve Lee, LACC Professor of Mathematics. Source: Steve Lee]

Solution:



$\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}=1$ ....(1)

$x-c=3(x_1-c)$ and $y=3y_1 \rightarrow x_1=\frac{x+2c}{3}$ and $y_1=\frac{y}{3}$ ....(2)

Plug (2) into (1), we get $\frac{\left(\frac{x+2c}{3}\right)^2}{a^2}+\frac{\left(\frac{y}{3}\right)^2}{b^2}=1 \rightarrow \frac{(x+2c)^2}{(3a)^2}+\frac{y^2}{(3b)^2}=1$

Therefore the locus of Q is the ellipse with center (-2c, 0), and the length of its major and minor axis are 3 times that of the given ellipse.

Problem 7) A set of three numbers are randomly drawn from the ten numbers 1, 2, 3, ..., 10. Find the probability that the set has one number divisible by 2, another number divisible by 3, and the other number divisible by 4.

[Problem submitted by Steve Lee, LACC Professor of Mathematics. Source: Steve Lee]

Solution:
If 4 and 8 are in the set that satisfies the conditions, then the other number may be 3, 6, or 9. Therefore we get the following 3 sets: $\{4,8,3\},\{4,8,6\},\{4,8,9\}$.

If 4 is in the set and 8 is not, then 4 must be the number divisible by 4, and the number divisible by 2 may be 2, 6, or 10: When the number divisible by 2 is 2, we get 3 sets: $\{4,2,3\},\{4,2,6\},\{4,2,9\}$.

When the number divisible by 2 is 6, we get 2 sets: $\{4,6,3\},\{4,6,9\}$.

When the number divisible by 2 is 10, we get 3 sets: $\{4,10,3\},\{4,10,6\},\{4,10,9\}$.

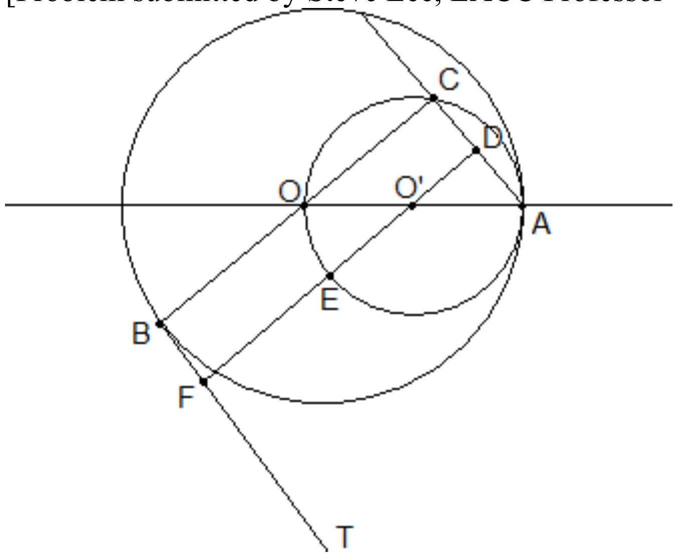
Similarly if 8 is in the set and 4 is not, then 8 must be the number divisible by 4, and the number divisible by 2 may be 2, 6, or 10: we get $\{8,2,3\},\{8,2,6\},\{8,2,9\}$; $\{8,6,3\},\{8,6,9\}$; $\{8,10,3\},\{8,10,6\},\{8,10,9\}$.

Therefore the total number of sets that satisfy the given conditions is 19.

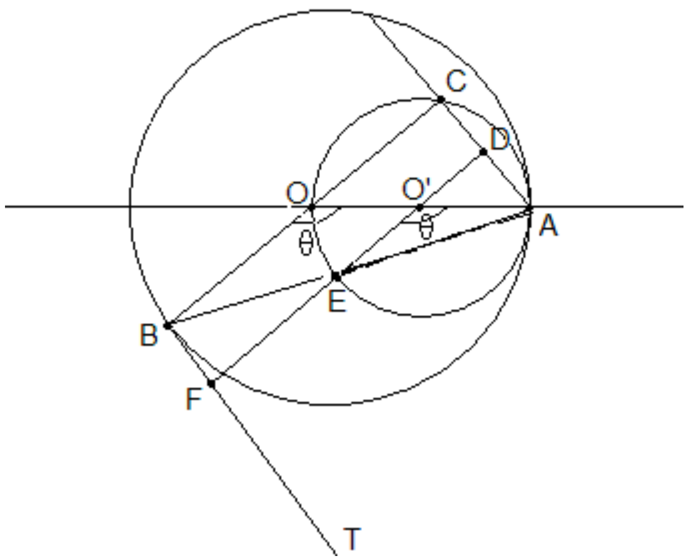
$$\therefore P = \frac{19}{{}_{10}C_3} = \frac{19}{120}.$$

Problem 8) In the following figure, circle O' is tangent to circle O at A and passing through the center of circle O. $\overline{BT}$ is a tangent line at an arbitrary point B of circle O. The line connecting B and O intersects circle O' at C. Draw $\overline{O'D}$ perpendicular to $\overline{AC}$ and extend the line to intersect circle O' at E, and $\overline{BT}$ at F. Prove $\overline{DE} \cong \overline{EF}$.

[Problem submitted by Steve Lee, LACC Professor of Mathematics. Source: Steve Lee]



Solution:
Connect points A and E. And connect points A and B. We need to prove that A, E, and B are collinear.



$\overline{OC} \perp \overline{AC}$, since the arc facing $\angle OCA$ is a half circle.

$\overline{O'D} \perp \overline{AC}$ $\therefore \overline{OC} \parallel \overline{O'D}$ $\therefore$ m$\angle$ AOB=m$\angle$ AO'E=θ

$\therefore$ m$\angle$ O'AE=$\frac{1}{2}(\pi-\theta)$, and m$\angle$ OAB=$\frac{1}{2}(\pi-\theta)$

$\therefore$ m$\angle$ O'AE= m$\angle$ OAB. $\therefore$ A,E, and B are collinear.

$\overline{OC} \parallel \overline{O'D}$ and AO'=$\frac{1}{2}$AO $\rightarrow$ AD=$\frac{1}{2}$AC $\rightarrow$ DE=$\frac{1}{2}$CB.

$\therefore$ DE=$\frac{1}{2}$DF, since CB=DF.

$\therefore$ EF=DF-DE=DF-$\frac{1}{2}$DF=$\frac{1}{2}$DF

$\therefore$ DE=EF

Problem 9) Given $\ln\left(1+\frac{1}{n}\right)<\frac{1}{n}$ for any positive integer n, prove $\left(\frac{n}{e}\right)^n < n!$ by using the method of mathematical induction.

[Problem submitted by Steve Lee, LACC Professor of Mathematics. Source: Steve Lee.]

Solution:

The inequality $\left(\frac{n}{e}\right)^n < n!$ is obviously true for n=1.

Assume $\left(\frac{n}{e}\right)^n < n!$. We need to show that $\left(\frac{n+1}{e}\right)^{n+1} < (n+1)!$ which is equivalent to $\frac{(n+1)^n}{e^{n+1}} < n!$....(1)

$$\ln\left(1+\frac{1}{n}\right)<\frac{1}{n} \rightarrow n\ln\left(1+\frac{1}{n}\right)<1 \rightarrow \ln\left(1+\frac{1}{n}\right)^n<1 \rightarrow \left(\frac{n+1}{n}\right)^n<e$$

$$\therefore (n+1)^n < e\cdot n^n$$

Divide both sides of the last inequality by e^{n+1}, we get

$\therefore \frac{(\mathrm{n}+1)^n}{\mathrm{e}^{\mathrm{n}+1}} < \frac{e\cdot n^n}{e^{n+1}} = \left(\frac{n}{e}\right)^n < n!$ $\therefore$ (1) is proved.

Multiply both sides of (1) by (n+1), we get $\left(\frac{n+1}{e}\right)^{n+1} < (n+1)!$

Problem 10) Find an integer solution for the linear equation $2011x + 2001y = 2$.

[Problem submitted by Steve Lee, LACC Professor of Mathematics. Source: Steve Lee.]

Solution:

2011=2001+10 $\rightarrow$ 10=2011-2001..(1)

2001=200 $\cdot$ 10+1 $\rightarrow$ 2001-200 $\cdot$ 10=1...(2)

Plug (1) into (2), we get 2001-200 $\cdot$ (2011-2001)=1. By rearranging the terms, we get

2011 $\cdot$ (-200) +2001 $\cdot$ 201=1..(3)

Multiply (3) by 2, we get 2011 $\cdot$ (-400) +2001 $\cdot$ 402=2.

Therefore **(-400, 402)** is an integer solution.

Note: It is easy to verify that (**-400**+2001n**, 402**-2011n) is the general solution, where n is any integer.