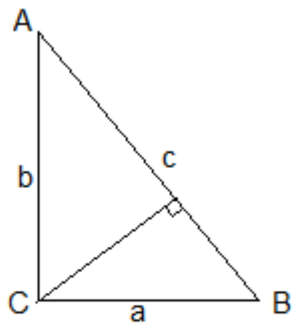
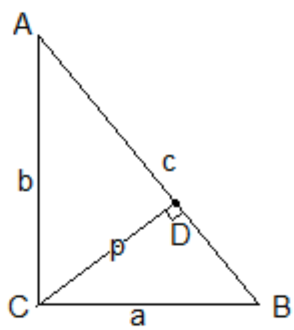
Problem 1) Suppose A, B, and C are the vertices of a right triangle with C being the vertex of the right angle, c the length of the hypotenuse, a the length of the leg opposite A, and b the length of the side opposite B. If a perpendicular is drawn from C to the hypotenuse, find its length in terms of a and b.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary's College Mathematics Contest Problems, Creative Publications, Inc. 1972.]



Solution:



Let D be the point of intersection of the perpendicular and the hypotenuse of triangle ABC. Let p be the length of the perpendicular. Consider triangle ABC and triangle DBC. Both are right triangles and both contain the angle whose vertex is B. So, they are similar triangles. Therefore,

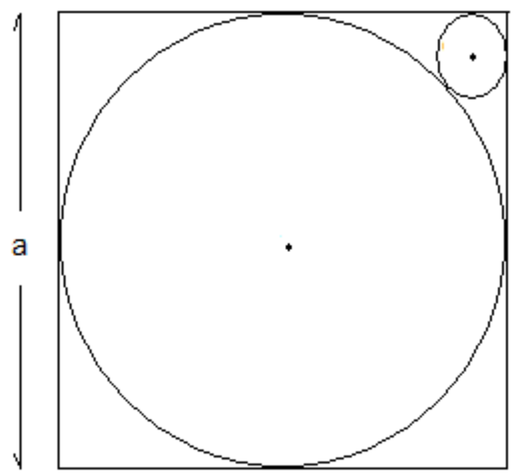
$$\frac{p}{a} = \frac{b}{c}$$

$$p = \frac{ab}{c}$$

$$p = \frac{ab}{\sqrt{a^2 + b^2}}$$

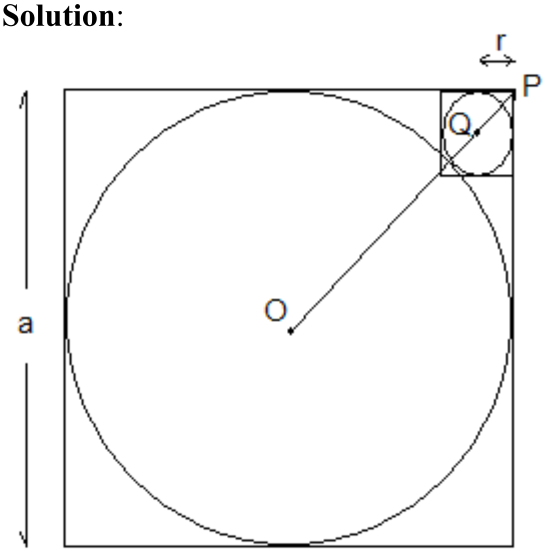
Problem 2) If a circle O of radius $\frac{a}{2}$ is inscribed in a square of side a, what is the radius of a circle Q that can be inscribed in a corner of the square tangent to two sides of the square and to the circle inscribed in the square?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary's College Mathematics Contest Problems, Creative Publications Inc. 1972.]



Solution: (next page)

Solution:



Let O be the center of the inscribed circle of radius $\frac{a}{2}$ and P the vertex of the square where the smaller circle is inscribed. Let r be the radius of the smaller circle with center Q. The length of the line segment OP is half the length of a diagonal of the square of side a; that is, $\frac{\sqrt{2}a}{2}$. The length of OP is also equal to $\frac{a}{2}$ plus r plus the length of the line segment QP. Now construct a square of side 2r with P being one of its vertices and circumscribing the smaller circle of radius r. The length of the line segment QP is half the length of a diagonal of the square of radius 2r. So, the length of QP is $\sqrt{2}r$. Therefore,

$$\frac{\sqrt{2}a}{2} = \frac{a}{2} + r + \sqrt{2}r$$

$$\sqrt{2}r + r = \frac{\sqrt{2}a}{2} - \frac{a}{2}$$

$$(\sqrt{2}+1)r = \frac{(\sqrt{2}-1)a}{2}$$

$$r = \frac{(\sqrt{2}-1)a}{2(\sqrt{2}+1)}$$

$$r = \frac{(\sqrt{2}-1)(\sqrt{2}-1)a}{2(\sqrt{2}+1)(\sqrt{2}-1)}$$

$$r = \frac{3-2\sqrt{2}}{2}a$$

Problem 3) Suppose S is the set of all points in the plane whose distance from the point (3,-1) is twice the distance from (4,2). Give a geometric description of S.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: February 2001 AMATYC competition.]

Solution: Let $(x, y) \in S$. Then the distance of (x,y) from (3,-1) is twice the distance of (x,y) from (4,2).

$$\sqrt{(x-3)^2 + (y+1)^2} = 2\sqrt{(x-4)^2 + (y-2)^2}$$

$$(x-3)^2 + (y+1)^2 = 4[(x-4)^2 + (y-2)^2]$$

$$3x^2 - 26x + 3y^2 - 18y = -70$$

$$x^2 - \frac{26}{3}x + y^2 - 6y = -\frac{70}{3}$$

$$x^2 - \frac{26}{3}x + \frac{169}{9} + y^2 - 6y + 9 = -\frac{70}{3} + \frac{169}{9} + 9$$

$$(x - \frac{13}{3})^2 + (y-3)^2 = \frac{40}{9}$$

This last equation implies that S is a circle of radius $\frac{2\sqrt{10}}{3}$ (approximately 2.1) centered at $(\frac{13}{3}, 3)$.

Problem 4) Five pirates were stranded on a desert island with their treasure. They became suspicious of each other. In order to be sure he would get his fair share, the first pirate counted out the coins and found that if he divided the total by 5 there were 4 coins left over. So, he took his approximate one fifth and an extra coin. Each pirate in turn did likewise and at each stage found exactly the same situation. Eventually they came to divide the treasure officially and found 1023 coins in the treasure chest. Each pirate noticed that the treasure was not intact, but since they had all done the same thing, no one indicated that anything was amiss. How many coins were there originally?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary's College Mathematics Contest Problems, Creative Publications, Inc. 1972.]

Solution: Let x_5 be one less than the number of coins the fifth pirate took. Then $4x_5 + 3 = 1023$. So, $x_5 = 255$ and the number of coins in the treasure chest before he took any was $5x_5 + 4 = 1279$. Let x_4 be one less than the number of coins the fourth pirate took. So, $4x_4 + 3 = 1279$. So, $x_4 = 319$ and the number of coins in the treasure chest before he took any was $5x_4 + 4 = 1599$. Continue this process to find that $x_3 = 399$, and before the third pirate took any coins there were 1999 coins in the treasure chest. Then $x_2 = 499$, and before the second pirate took any coins there were 2499 coins in the treasure chest. And finally $x_1 = 624$, and originally before any of the pirates had taken any coins there were 3124 coins in the treasure chest.

Problem 5) Find all pairs of real numbers (x, y) such that $\frac{\log_2 x}{\log_2 y} = \log_2 \frac{x}{y}$.

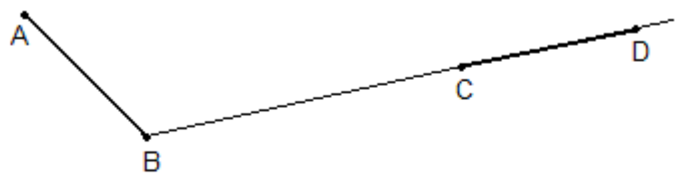
[Problem submitted by LACC Professor of Mathematics, Vin Lee. Source: Vin Lee.]

Solution: Note that $\frac{\log_2 x}{\log_2 y} = \log_2 x - \log_2 y$. Substitute $h = \log_2 x,\ \ k = \log_2 y$ into this equation to get $\frac{h}{k} = h - k$ and $k^2 - hk + h = 0$. Now use the quadratic formula to solve this equation for k in terms of h resulting in $k = \frac{h \pm \sqrt{h^2 - 4h}}{2}$. Note that $x = 2^h$ and $y = 2^k$. So, for y to be real k must be real which implies $h^2 - 4h \geq 0$. Solving this last inequality, we get $h \leq 0$ or $h \geq 4$. However, if $h = 0$, then $k = 0$ and $y = 1$ causing the denominator on the left side of the original equation to be zero. So, we conclude that $h < 0$ or $h \geq 4$.

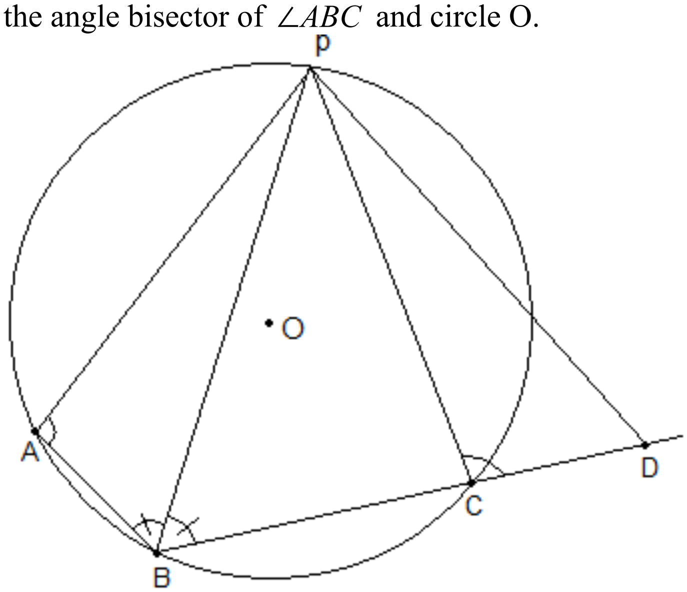
Therefore, the answer to the question is $(x, y) = (2^h, 2^k)$ with $h < 0$ or $h \geq 4$ and $k = \frac{h \pm \sqrt{h^2 - 4h}}{2}$.

Problem 6) Line AB and line CD intersect at B, and $\overline{AB} = \overline{CD}$ (See the figure below.) Find a point P on the plane of the two lines such that $\Delta PAB \cong \Delta PCD$.

[Problem submitted by Steve Lee, LACC Professor Emeritus. Source: Steve Lee.]



Solution:
Construct the circumscribed circle O for points A, B, and C. Then P is the intersection of the angle bisector of $\angle ABC$ and circle O.



Proof:

$m\angle PBA = m\angle PBC \rightarrow \overline{PA} = \overline{PC}$

$m\angle PAB = m\angle PCD$

$\overline{AB} = \overline{CD}$

$\therefore \Delta PAB \cong \Delta PCD$

Problem 7) Suppose a sequence is defined for n = 1, 2, 3, … as $a_1 = 1$, $a_{2n} = 2a_{2n-1}$, and $a_{2n+1} = a_{2n} + 1$. Estimate $\sqrt[2012]{a_{2012}}$ (the two thousand and twelfth root of a_{2012}) to 3 decimal places.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: February 2001 AMATYC competition.]

Solution:

$$a_1 = 1$$
$$a_2 = 2a_1 = 2 \cdot 1 = 2$$
$$a_3 = a_2 + 1 = 2 + 1$$
$$a_4 = 2a_3 = 2 \cdot (2+1) = 2^2 + 2$$
$$a_5 = a_4 + 1 = 2^2 + 2 + 1$$
$$a_6 = 2a_5 = 2\left(2^2 + 2 + 1\right) = 2^3 + 2^2 + 2$$

Now consider the sequence of even terms:

$$a_2 = 2,$$
$$a_4 = 2^2 + 2,$$
$$a_6 = 2^3 + 2^2 + 2,$$
$$\ldots$$
$$a_{2n} = 2^n + 2^{n-1} \ldots\ldots + 2$$
$$= 2\left(2^{n-1} + 2^{n-2} \ldots + 1\right) = 2\left(2^n - 1\right) = 2^{n+1} - 2$$
$$a_{2n} = 2^{n+1} - 2 \rightarrow a_{2012} = 2^{1006+1} - 2 = 2^{1007} - 2.$$

$$\therefore \sqrt[2012]{a_{2012}} = \sqrt[2012]{2^{1007} - 2} \approx \sqrt[2012]{2^{1007}} = 2^{\frac{1007}{2012}} \approx 2^{\frac{1}{2}} \approx 1.414$$

Problem 8) Let a_1, a_2, a_3, and a_4 be the solutions of the equation $x^4 + \sqrt{3}\left(x^2 + x + 2\right) = 0$. Find the *exact* value of $a_1{}^4 + a_2{}^4 + a_3{}^4 + a_4{}^4$.

[Problem submitted by Steve Lee, LACC Professor Emeritus. Source: Steve Lee.]

Solution:

$x^4 + \sqrt{3}\left(x^2 + x + 2\right) = 0$ ....(1)

Since a_1, a_2, a_3, and a_4 are the solutions of equation (1), we get:

$a_1{}^4 + \sqrt{3}\left(a_1{}^2 + a_1 + 2\right) = 0$

$a_2{}^4 + \sqrt{3}\left(a_2{}^2 + a_2 + 2\right) = 0$

$a_3{}^4 + \sqrt{3}\left(a_3{}^2 + a_3 + 2\right) = 0$

$a_4{}^4 + \sqrt{3}\left(a_4{}^2 + a_4 + 2\right) = 0$

By adding the 4 equations above, we get

$\left(a_1{}^4 + a_2{}^4 + a_3{}^4 + a_4{}^4\right) + \sqrt{3}\left[\left(a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2\right) + \left(a_1 + a_2 + a_3 + a_4\right) + 8\right] = 0 \cdots\cdots\cdots (2)$

$\left(x - a_1\right)\left(x - a_2\right)\left(x - a_3\right)\left(x - a_4\right) = 0 \rightarrow$

$x^4 - \left(a_1 + a_2 + a_3 + a_4\right)x^3 + \left(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4\right)x^2$

$-\left(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4\right)x + a_1a_2a_3a_4 = 0$..(3)

Compare the coefficients of equation (1)and (3),

we get $a_1 + a_2 + a_3 + a_4 = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$

and $a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4 = \sqrt{3} \cdots\cdots\cdots\cdots\cdots\cdots (5)$

$a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2 = \left(a_1 + a_2 + a_3 + a_4\right)^2 - 2\left(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4\right) \cdots (6)$

Plug equation $(4),(5)$ into (6), we get $a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2 = -2\sqrt{3} \cdots (7)$

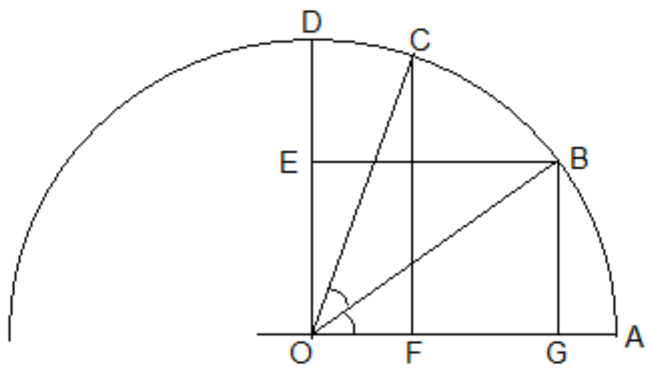
Plug equation (4) and (7) into equation (2), we get

$\left(a_1{}^4 + a_2{}^4 + a_3{}^4 + a_4{}^4\right) + \sqrt{3}\left[-2\sqrt{3} + 0 + 8\right] = 0$

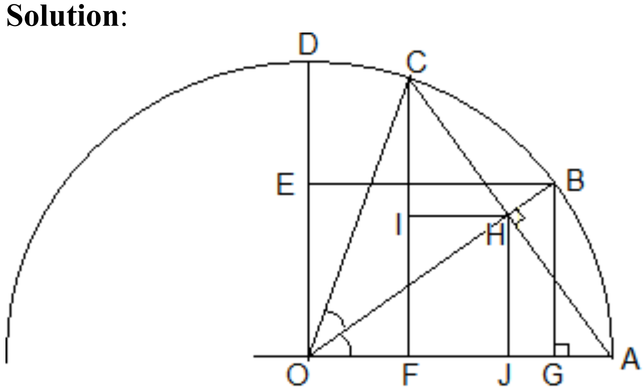
$\therefore a_1{}^4 + a_2{}^4 + a_3{}^4 + a_4{}^4 = 6 - 8\sqrt{3}$.

Problem 9) In the figure below, points A, B, C, and D are on the circle with center at O and radius = 1. DO, CF, and BG are perpendicular to OA, $m\angle AOB = m\angle BOC$. Prove that the length of $\overline{CF}$ = two times the area of the rectangle OGBE.

[Problem submitted by Roger Wolf, LACC Professor of Mathematics. Source: Roger Wolf.]



Solution:



Connect A and C. AC intersects OB at H. Then $AC \perp OB$ and $\overline{CH} = \overline{HA}$, since $m\angle AOB = m\angle BOC$.

Draw HI parallel to OA and intersecting CF at I. Draw HJ perpendicular to OA and intersecting OA at J. $\therefore \Delta OHJ \cong \Delta OBG$

$\frac{\overline{HJ}}{\overline{BG}} = \frac{\overline{OH}}{\overline{OB}} = \frac{\overline{OH}}{1} \rightarrow \overline{HJ} = \overline{OH} \cdot \overline{BG}$ ....(1)

$\overline{CH} = \overline{HA} \rightarrow \overline{HA} = \frac{1}{2}\overline{CA} \rightarrow \overline{IF} = \frac{1}{2}\overline{CF}$ since $\Delta CIH \cong \Delta CFA$.

Therefore $\overline{HJ} = \overline{IF} = \frac{1}{2}\overline{CF}$ ....(2)

$\Delta AOH \cong \Delta BOG \rightarrow \overline{OH} = \overline{OG}$ ....(3)

Substitute (2) and (3) into (1), we get

$\frac{1}{2}\overline{CF} = \overline{OG} \cdot \overline{BG} \rightarrow \overline{CF} = 2\overline{OG} \cdot \overline{BG}$.

Problem 10) Show that the product of three consecutive odd positive integers can not be equal to the kth power of a positive integer, where k is any integer greater than 1.

[Problem submitted by Steve Lee, LACC Professor Emeritus. Source: Steve Lee.]

Solution:

Suppose the opposite is true. Then there is a positive integer r such that n(n+2)(n+4) $= r^k$, where n is an odd positive integer; and k>1.

r can be decomposed in a unique way as $r = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$, $\left(1 < p_1 < p_2 \cdots < p_m\right)$, where each p_i is a prime number. $\therefore n(n+2)(n+4) = \left(p_1^{e_1}\right)^k \left(p_2^{e_2}\right)^k \cdots \left(p_m^{e_m}\right)^k$ ....(1)

Since $(n+2) - n = 2$, a common factor of n and $(n+2)$ must be a factor of 2. But 2 can not be a common factor, since n and $(n+2)$ are odd. Therefore n and $(n+2)$ are relatively prime. Similarly n and $(n+4)$ are relatively prime; and $(n+2)$ and $(n+4)$ are relatively prime. We have shown that the 3 integers $n, (n+2)$, and $(n+4)$ in equation (1) are relatively prime.

If some p_i in equation (1) is a factor of n, then it can not be a factor of $(n+2)$ or $(n+4)$. This implies that $\left(p_i^{e_i}\right)^k$, where $e_i k > 1$, must be a factor of n. Then all the factors of n are of the form $\left(p_i^{e_i}\right)^k$ which is the kth power of an integer. This implies that n is the kth power of some integer.

Similarly $(n+2)$ and $(n+4)$ are the kth power of some integers.

$\therefore\ n = s^k,\ (n+2) = t^k,\ (n+4) = u^k,$ where $u > t > s \geq 1$ are positive integers.

$$t > s \rightarrow t \geq (s+1) \rightarrow (n+2) = t^k \geq (s+1)^k = s^k + ks^{k-1} + \frac{k(k-1)}{2} s^{k-2} + \ldots + 1 > s^k + k \geq n+2$$

That is, $(n+2) > n+2$. We get a contradiction. Therefore the statement in the problem is true.