**Problem 1**) After Ed eats 20% of his pie and Ann eats 40% of her pie, Ed has twice as much pie left as Ann. Find Ed’s original amount of pie as a percentage of Ann’s original pie.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: 2011 AMATYC Test #2]

**Solution:** Let \( x \) be the original amount of Ed’s pie and \( y \) be the original amount of Ann’s pie. Then the first sentence of the problem may be written as this equation: \( .8x = 2(.6y) \).

So, \( y = \frac{2}{3} x \). Ed’s original amount of pie as a percentage of Ann’s original pie is

\[
\frac{x}{y} \times 100\% \quad \text{and} \quad \frac{x}{y} \times 100\% = \frac{x}{2} \times 100\% = \frac{2x}{3} \times 100\% = 150\%.
\]
Problem 2) Express $\frac{\sqrt{4 + 2\sqrt{3}} - \sqrt{28 + 10\sqrt{3}}}{15}$ as a rational number.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary’s College Mathematics Contest Problems for Junior and Senior High School by Brother Alfred Brousseau, 1972]

Solution: Note that $4 + 2\sqrt{3} = (1 + \sqrt{3})^2$ and $28 + 10\sqrt{3} = (5 + \sqrt{3})^2$. So,

$$\frac{\sqrt{4 + 2\sqrt{3}} - \sqrt{28 + 10\sqrt{3}}}{15} = \frac{1 + \sqrt{3} - (5 + \sqrt{3})}{15} = \frac{-4}{15}$$
**Problem 3)** If b varies over all real numbers, upon what curve do the vertices of the parabolas with equations \( f(x) = x^2 + bx + 2 \) lie?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: 1989 AMATYC Exam #1]

**Solution:** The x-coordinate of the vertex of \( f(x) = x^2 + bx + 2 \) is i) \( x = -\frac{b}{2} \). The y-coordinate of the vertex is \( y = f\left(-\frac{b}{2}\right) \). So, ii) \( y = 2 - \frac{b^2}{4} \). From i) \( b = -2x \). Substitute into ii) to get \( y = 2 - x^2 \). The graph of this equation, a downward opening parabola whose vertex is (0,2), is the answer to the question.
Problem 4) The volumes of two cubes differ by 259 cm³. If the edges of one cube are each 4 cm greater than the edges of the other, find sum of the lengths of one edge of each cube.

[Problem submitted by Anatoliy Nikolaychuk, LACC Professor of Mathematics. Source: February 2000 AMATYC]

Solution: \( (x + 4)^3 - x^3 = 259 \)
\[ x^2 + 16x - 65 = 0 \]
\[ (2x - 5)(2x + 13) = 0 \]
\[ x = \frac{5}{2}, \quad x + 4 = \frac{13}{2} \]
\[ x + (x + 4) = \frac{5}{2} + \frac{13}{2} = 9 \]
Problem 5) Sand is pouring from a funnel onto a cone-shaped pile which is 20 feet high and 50 feet in diameter at its base. If the sand is coming out of the funnel at a rate of 2.5 cubic feet per second, how long will it take for the height of the pile to increase 1 inch, assuming that the shape of the pile does not change?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary’s College Mathematics Contest Problems for Junior and Senior High School by Brother Alfred Brousseau, 1972]

Solution: Let $x$ be the change in the radius of the base. Then

$$\frac{20}{25} = \frac{20 + \frac{1}{12}}{25 + x}$$

$$x = \frac{5}{48}.$$  

The volume of a cone is $V = \frac{1}{3} \pi r^2 h$.

The change in volume is $\Delta V = \frac{\pi}{3} (25 + \frac{5}{48})^2 (20 + \frac{1}{12}) - \frac{\pi}{3} \cdot 25^2 \cdot 20 = 164.3$. 

$$\text{time} = \frac{164.3}{2.5} = 65.7 \text{ seconds or } \text{time} = 20.9\pi \text{ seconds}$$
Problem 6) Suppose $x$ and $f(x)$ are real numbers. Find the inverse function of $f(x) = x + \sqrt{x}$. That is, find a function $f^{-1}(x)$ such that $f^{-1}[f(x)] = x$ for every $x$ in the domain of $f(x)$. Prove your answer is correct.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: Let $y = f(x)$ and solve $y = x + \sqrt{x}$ for $x$:

$$0 = x + \sqrt{x} - y.$$ Use the quadratic formula to get

$$\sqrt{x} = \frac{-1 \pm \sqrt{1 + 4y}}{2}.$$ So,

$$x = \left(\frac{-1 \pm \sqrt{1 + 4y}}{2}\right)^2.$$ Since $f(x)$ is a one-to-one function, its inverse is a function. This gives two possibilities for $f^{-1}(x)$. Either

A) $f^{-1}(x) = \left(\frac{-1 - \sqrt{1 + 4x}}{2}\right)^2$ or B) $f^{-1}(x) = \left(\frac{-1 + \sqrt{1 + 4x}}{2}\right)^2$, but not both. Consider the first possibility:

A) $f^{-1}(x) = \left(\frac{-1 - \sqrt{1 + 4x}}{2}\right)^2$

$$= \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4x} + x.$$ Then

$$f^{-1}[f(x)] = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4(x + \sqrt{x})} + x + \sqrt{x}$$

$$= \frac{1}{2} + \frac{1}{2} \sqrt{(2\sqrt{x} + 1)^2} + x + \sqrt{x}$$

$$= x + 2\sqrt{x} + 1$$

$\neq x$. So, A) is not the inverse function. Now consider the second possibility:

B) $f^{-1}(x) = \left(\frac{-1 + \sqrt{1 + 4x}}{2}\right)^2$

$$= \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4x} + x.$$ Then

$$f^{-1}[f(x)] = \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4(x + \sqrt{x})} + x + \sqrt{x}$$

$$= \frac{1}{2} - \frac{1}{2} \sqrt{(2\sqrt{x} + 1)^2} + x + \sqrt{x}$$

$$= x.$$ Therefore, B) is the inverse function.
Problem 7) Find the sum: $\sum_{k=1}^{\infty} \frac{k}{8^k}$.

[Problem submitted by Anatoliy Nikolaychuk, LACC Professor of Mathematics. Source: February 2000 AMATYC]

Solution: Let $S = \sum_{k=1}^{\infty} \frac{k}{8^k}$. Then

\[
S = \frac{1}{8} + \frac{2}{8^2} + \frac{3}{8^3} + \frac{4}{8^4} + \cdots
\]

\[
S = \frac{1}{8^2} + \frac{2}{8^3} + \frac{3}{8^4} + \cdots
\]

\[
S - S = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \frac{1}{8^4} + \cdots
\]

\[
\frac{7}{8} S = \sum_{k=1}^{\infty} \frac{1}{8^k}
\]

\[
\frac{7}{8} S = \frac{1}{8 - \frac{1}{8}}
\]

\[
S = \frac{8}{49}
\]
**Problem 8)** Suppose \( x \geq 0 \). Find the inverse function of \( f(x) = 2^{x-1} + 2^{-(x+1)} \). That is, find a function \( f^{-1}(x) \) such that \( f^{-1}[f(x)] = x \) for every \( x \) in the domain of \( f(x) \). Also, find the domain and range of \( f^{-1}(x) \).

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

**Solution:** Note that the domain of \( f(x) \) is \([0, \infty)\). \( f(x) \) is increasing and \( f(0) = 1 \).

[By plotting a few points of \( f(x) \) it can be seen to be increasing or it can be proved to be increasing on \((1, \infty)\) by taking its derivative.] So, the range of \( f(x) \) is \([1, \infty)\).

Therefore, the domain of \( f^{-1}(x) \) is \([1, \infty)\) and its range is \([0, \infty)\). Let \( y = f(x) \) and solve \( y = 2^{x-1} + 2^{-(x+1)} \) for \( x \). First multiply both sides of the equation by \( 2^{x+1} \) to get

\[
2y2^x = 2^{2x} + 2^0
\]

\[
0 = 2^{2x} - 2y2^x + 1.
\]

Use the quadratic formula to get

\[
x = \frac{2y \pm \sqrt{4y^2 - 4}}{2}
\]

\[
x = y \pm \sqrt{y^2 - 1}.
\]

Because \( x \geq 0, 2^x \geq 1 \). However, \( y - \sqrt{y^2 - 1} \) is not greater than or equal to 1 for every \( y \in [1, \infty) \). For example

\[
2 - \sqrt{2^2 - 1} = 2 - \sqrt{3} = .27.
\]

Therefore,

\[
x = \log_2(y + \sqrt{y^2 - 1}).
\]

Substitute \( f^{-1}(x) \) for \( x \) and \( x \) for \( y \) to get

\[
f^{-1}(x) = \log_2(x + \sqrt{x^2 - 1}).
\]

* Or prove that \( y - \sqrt{y^2 - 1} < 1 \) for every \( y \in (1, \infty) \):

\[
1 < y
\]

\[
1 - y < 0
\]

\[
2 - 2y < 0
\]

\[
1 - 2y < -1
\]

\[
1 - 2y + y^2 < -1 + y^2
\]

\[
(y - 1)^2 < y^2 - 1
\]

\[
y - 1 < \sqrt{y^2 - 1}
\]

\[
y - \sqrt{y^2 - 1} < 1
\]
Problem 9) In the equation \( x^4 - 4x^3 - 12x^2 - 13x + 20 = 0 \) what is the sum of the squares of the four roots (solutions)?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Saint Mary’s College Mathematics Contest Problems for Junior and Senior High School by Brother Alfred Brousseau, 1972]

Solution: Let \( a, b, c, \) and \( d \) be the roots of the equation. Then
\[
(x - a)(x - b)(x - c)(x - d) = x^4 - 4x^3 - 12x^2 - 13x + 20.
\]
Now multiply to get the following.
\[
\begin{align*}
(x - a)(x - b) &= x^2 - (a + b)x + ab \\
(x - c)(x - d) &= x^2 - (c + d)x + cd
\end{align*}
\]
Next multiply \([x^2 - (a + b)x + ab][x^2 - (c + d)x + cd]\) to get
\[
x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 - [ab(c + d) + cd(a + b)]x + abcd
\]
Equating the third and second degree coefficients in this equation with those in the equation given in the problem, we get \( a + b + c + d = 4 \) and \( ab + ac + ad + bc + bd + cd = -12 \). These two values will be substituted into an equation below.

Consider \( (a + b + c + d)^2 \). Expand to get this equation:
\[
(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd).
\]
So, \( 4^2 = a^2 + b^2 + c^2 + d^2 + 2(-12) \) and \( a^2 + b^2 + c^2 + d^2 = 40 \). 
**Problem 10** Let $\hat{\Diamond}$ be an operation (like addition or multiplication) which associates each pair $x,y$ of real numbers with the real number $x \hat{\Diamond} y$ such that for all real numbers $x$, $y$, $z$ the following conditions are satisfied: 1) $x \hat{\Diamond} x = x$, 2) $x \hat{\Diamond} y = y \hat{\Diamond} x$, 3) $x \hat{\Diamond} (y \hat{\Diamond} z) = (x \hat{\Diamond} y) \hat{\Diamond} z$, and 4) if $y < z$ and $x \hat{\Diamond} y \neq x$ then $x \hat{\Diamond} y < x \hat{\Diamond} z$. Prove that for every pair of real numbers $x,y$ $x \hat{\Diamond} y = x$ or $x \hat{\Diamond} y = y$. Also, find an operation that satisfies these four conditions.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: 1994-95 Pomona-Wisconsin Talent Search, Problem Set IV]

**Solution**: Assume that there exists a pair of real numbers $x$ and $y$ such that $x \hat{\Diamond} y \neq x$ and $x \hat{\Diamond} y \neq y$. From condition 1) we know $x \neq y$. Without loss of generality say $y < x$. Then from conditions 1) and 4) we can conclude that $x \hat{\Diamond} y < x \hat{\Diamond} x = x$; that is, A) $x \hat{\Diamond} y < x$.

Now using conditions 1), 2), and 3) we conclude that $y \hat{\Diamond} (x \hat{\Diamond} y) = (y \hat{\Diamond} x) \hat{\Diamond} y = x \hat{\Diamond} (y \hat{\Diamond} y) = x \hat{\Diamond} y$; that is, B) $y \hat{\Diamond} (x \hat{\Diamond} y) = x \hat{\Diamond} y$.

Next using our assumption $x \hat{\Diamond} y \neq y$, A), B), and condition 4) we get that $x \hat{\Diamond} y < x$ and $y \hat{\Diamond} (x \hat{\Diamond} y) \neq y$ implies that $y \hat{\Diamond} (x \hat{\Diamond} y) < y \hat{\Diamond} x$. Then according to B) and condition 1) $x \hat{\Diamond} y < x \hat{\Diamond} y$ which is a contradiction.

So, our original assumption that there exists a pair of real numbers $x,y$ such that $x \hat{\Diamond} y \neq x$ and $x \hat{\Diamond} y \neq y$ must be false. Therefore, for every pair of real numbers $x,y$ $x \hat{\Diamond} y = x$ or $x \hat{\Diamond} y = y$.

Two operations satisfying all four of the given conditions are $\max\{x,y\}$ and $\min\{x,y\}$.