Problem 1)
The sum of three numbers is 27. The largest minus the smallest is 6. The second largest minus the smallest is 3. What are the three numbers?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: Let $x$ be the largest number, $y$ the second largest, and $z$ the smallest. The first three sentences in the problem may be written as three equations:

\[
\begin{align*}
x + y + z &= 27 \\
x - z &= 6 \\
y - z &= 3
\end{align*}
\]

Note that $\begin{align*} x - z &= 6 &\rightarrow& & x = z + 6 \\
y - z &= 3 &\rightarrow& & y = z + 3 \end{align*}$

Substitute these values into the first equation to get $(z + 6) + (z + 3) + z = 27$. Then solve to get $z = 6, \ y = 9, \ and \ x = 12$. 
Problem 2)
Find the ratio of the area a circle to the area of an inscribed square.
[Problem submitted by Kee Lam, LACC Professor of Mathematics. Source: Kee Lam]

Solution: Let the measure of the side of the square be a. Since the square is inscribed in the circle, the diagonal of the square is 2r. Using the Pythagorean Theorem and solve for \( r^2 \) terms of a,

\[
a^2 + a^2 = (2r)^2
\]

\[
2a^2 = 4r^2
\]

\[
\frac{2a^2}{4} = r^2
\]

\[
r^2 = \frac{a^2}{2}
\]

Then, the area of the circle is \( \pi r^2 = \frac{\pi a^2}{2} \). Hence, the ratio of the area a circle to the area of an inscribed square is \( \frac{\pi a^2}{2} : a^2 \), which is \( \frac{\pi}{2} : 1 \).
Problem 3)
A sphere is inscribed within a cube which has a surface area of 24 square meters. A second cube is inscribed within the sphere. What is the surface area of the inner cube?


Solution: Let $e_1$ be the length of each edge of the outer cube. Then $24 = 6e_1^2 \rightarrow e_1^2 = 4$.

Then the diameter of the sphere inscribed within this cube must be the same as the length of each edge of the cube, $e_1 = 2$.

Now let $d$ be the length of the diagonal of the cube inscribed within this sphere. Then $d = 2$.

Let $e_2$ be the length of each edge of the inner cube. Then $d^2 = e_2^2 + (\sqrt{2}e_2)^2 \rightarrow e_2^2 = \frac{4}{3}$.

The surface area of this cube is 8 square meters.
Problem 4)
The first four terms of an arithmetic sequence are $p, 9, 3p - q, \text{ and } 3p + q$. What is the 2016th term of this sequence?

[Problem submitted by Ha Nguyen, LACC Professor of Mathematics. Source: 2010 AMC, Problem 10]

Solution: Let $d$ be the common difference between consecutive terms. Then

\[
\begin{align*}
    d &= 9 - p \\
    d &= (3p - q) - 9 \\
    d &= (3p + q) - (3p - q) \\
    &\rightarrow \quad d = 3p - q - 9 \\
    &\rightarrow \quad d = 2q
\end{align*}
\]

Use these equations to find $p = 5$ and $q = 2$. So, the first term is 5 the common difference is 4. Therefore, the $n$th term is $a_n = 5 + (n - 1)4 = 1 + 4n$. The 2016th term is $1 + 4 \cdot 2016 = 8065$
Problem 5)

Find the sum \( \sum_{n=0}^{50} \frac{1}{4n^2 - 1} \)

[Problem submitted by Kee Lam, LACC Professor of Mathematics.  Source: Kee Lam]

Solution: Let’s decompose the fraction of \( \frac{1}{4n^2 - 1} \). That is,

\[
\frac{1}{4n^2 - 1} = \frac{A}{2n + 1} + \frac{B}{2n - 1}
\]

\[
\frac{1}{4n^2 - 1} = \frac{A(2n - 1) + B(2n + 1)}{(2n + 1)(2n - 1)}
\]

\[
\frac{1}{4n^2 - 1} = \frac{2An - A + 2Bn + B}{(2n + 1)(2n - 1)}
\]

Hence, \( 2An + 2Bn = (2A + 2B)n = 0n \) implies \( 2A + 2B = 0 \) and \( -A + B = 1 \). Solving the system of \( 2A + 2B = 0 \) and \( -A + B = 1 \), we get \( A = -\frac{1}{2} \) and \( B = \frac{1}{2} \).

Then,

\[
\sum_{n=0}^{50} \frac{1}{4n^2 - 1} = \sum_{n=0}^{50} \left[ -\frac{1}{2} \left( \frac{1}{2n + 1} \right) + \frac{1}{2} \left( \frac{1}{2n - 1} \right) \right]
\]

\[
= \sum_{n=0}^{50} \frac{1}{2} \left[ \frac{1}{2n - 1} - \frac{1}{2n + 1} \right]
\]

\[
= \frac{1}{2} \sum_{n=0}^{50} \left[ \frac{1}{2n - 1} - \frac{1}{2n + 1} \right]
\]

\[
= \frac{1}{2} \left( -1 - \frac{1}{101} \right)
\]

\[
= -\frac{51}{101}
\]
Problem 6)
Find $k$ such that the sum of the squares of the roots of $x^2 + 2x + k = 0$ would be equal to 10.

[Problem submitted by Kian Kaviani, LACC Mathematics Department Chairman. Source: Kian Kaviani]

$$x_1 + x_2 = 2$$
$$\left(x_1 + x_2\right)^2 = 4$$

Solution: $$(x_1)^2 + (x_2)^2 + 2x_1x_2 = 4$$
$$10 + 2k = 4$$
$$k = -3$$
Problem 7)

Suppose a line whose slope is \( \sqrt{3} \) is drawn through the focus F of the parabola \( y^2 = 8(x + 2) \). If the two points of intersection of the line and the parabola are A and B, and the perpendicular bisector of the chord AB intersects the x-axis at point P, what is the length of the segment PF?


Solution: The equation is given in the standard form \((y - k)^2 = 4a(x - h)\) of a parabola opening to the right with its vertex at \((h, k)\) and distance to the focus \(a\). The vertex of this parabola is \((-2, 0)\) and the focus F \((0, 0)\). So, the equation of the line through F with slope \( \sqrt{3} \) is \( y = \sqrt{3}x \). To find the x-coordinates of the points of intersection of this line with the parabola, substitute \( \sqrt{3} \) for \( y \) in \( y^2 = 8(x + 2) \):

\[
3x^2 - 8x - 16 = 0
\]

\[
x = \frac{8 \pm \sqrt{64 + 192}}{6}
\]

\[
x = \frac{4 \pm 8}{3}
\]

\[
x = \frac{4}{3}, 4
\]

The x-coordinate of the midpoint of the chord AB is \( x = \frac{1}{2} \left( \frac{-4}{3} + 4 \right) = \frac{4}{3} \). The y-coordinate is \( \frac{4}{\sqrt{3}} \). The distance from the focus to the midpoint is \( \sqrt{\left( \frac{4}{3} \right)^2 + \left( \frac{4}{\sqrt{3}} \right)^2} = \frac{8}{3} \)

Let E denote the midpoint. Note that the segments FE, EP, and FP form a 30, 60, 90 degree triangle whose legs are FE and EP and whose hypotenuse is FP. The hypotenuse is twice the length of the shorter leg, \( \frac{8}{3} \cdot 2 = \frac{16}{3} \), which is the answer to the question.
Problem 8)

Suppose \( \log_4(x + 2y) + \log_4(x - 2y) = 1 \). Find the minimum value of \(|x| - |y|\). [Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Problem 11, page 8 of Mathematical Olympiad in China by Xiong Bin and Lee Peng Yee, East China Normal University Press, 2007.]

Solution: Since the domain of the log function is \((0, \infty)\), \(x + 2y > 0\) and \(x - 2y > 0\). Adding these inequalities shows \(x > 0\). Also, the second inequality indicates \(y < \frac{1}{2}x\).

The equation may be rewritten as \( \log_4(x + 2y)(x - 2y) = 1 \) which implies
\[
(x + 2y)(x - 2y) = 4
\]
\[
x^2 - 4y^2 = 4
\]
\[
x^2 - \frac{y^2}{1^2} = 1
\]
This is the standard form of the equation of a hyperbola centered at the origin with vertices \((\pm 2, 0)\) and asymptotes \(y = \pm \frac{1}{2}x\). However, since \(x > 0\), the graph of the equation is only the right side of the hyperbola with the single vertex \((2, 0)\).

Since the graph is symmetric about the x-axis, without loss of generality we may consider only the case of \(y \geq 0\) for which \(|x| - |y| = x - y\). Let \(u = x - y\) and substitute \(u + y\) for \(x\) in the equation \(x^2 - 4y^2 = 4\) to get
\[
(u + y)^2 - 4y^2 = 4
\]
\[
u^2 + 2uy + y^2 - 4y^2 = 4
\]
\[
u^2 + 2uy - 3y^2 = 4
\]
\[
3y^2 - 2uy + 4 - u^2 = 0
\]
Now use the quadratic formula to solve this equation for \(y\) in terms of \(u\).
\[
y = \frac{2u \pm \sqrt{4u^2 - 12(4 - u^2)}}{6}
\]
\[
y = \frac{u \pm \sqrt{u^2 - 3}}{3}
\]
This equation has real solutions if and only if \(u \leq -\sqrt{3}\) or \(u \geq \sqrt{3}\).

Recall from the first paragraph of this solution that \(y < \frac{1}{2}x\). Since \(x > \frac{1}{2}x > y\) and \(x > 0\), \(x - y > 0\). Therefore, \(u \geq \sqrt{3}\). So, \(\sqrt{3}\) is the minimum value of \(|x| - |y|\).
Problem 9)
How many integers from 1000 to 2016 when tripled have no even digits?

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: February/March 2015 AMATYC Student Mathematics League]

Solution: Note that when multiplying each of the integers from 1000 through 2016 by 3, all of the products are 4 digit numbers whose first digit is 3, 4, 5, or 6. Since we are asked how many of these products have no even digits, we need only consider those whose first digit is 3 or 5.

Using the fact that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3, for 4 digit integers with odd digits which are divisible by 3 and whose first digit is 3, the possibilities for the last 3 digits are:

<table>
<thead>
<tr>
<th>Last 3 digits</th>
<th>Permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
<td>1</td>
</tr>
<tr>
<td>117</td>
<td>3</td>
</tr>
<tr>
<td>177</td>
<td>3</td>
</tr>
<tr>
<td>135</td>
<td>6</td>
</tr>
<tr>
<td>159</td>
<td>6</td>
</tr>
<tr>
<td>333</td>
<td>1</td>
</tr>
<tr>
<td>337</td>
<td>6</td>
</tr>
<tr>
<td>339</td>
<td>3</td>
</tr>
<tr>
<td>357</td>
<td>6</td>
</tr>
<tr>
<td>359</td>
<td>3</td>
</tr>
<tr>
<td>399</td>
<td>3</td>
</tr>
<tr>
<td>555</td>
<td>1</td>
</tr>
<tr>
<td>579</td>
<td>6</td>
</tr>
<tr>
<td>777</td>
<td>1</td>
</tr>
<tr>
<td>999</td>
<td>1</td>
</tr>
</tbody>
</table>

This is a total of 41 numbers.

Now consider the last 3 digits of positive 4 digit integers with odd digits which are divisible by 3 and whose first digit is 5:

<table>
<thead>
<tr>
<th>Last 3 digits</th>
<th>Permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>115</td>
<td>3</td>
</tr>
<tr>
<td>133</td>
<td>3</td>
</tr>
<tr>
<td>139</td>
<td>6</td>
</tr>
<tr>
<td>157</td>
<td>6</td>
</tr>
<tr>
<td>199</td>
<td>3</td>
</tr>
<tr>
<td>337</td>
<td>3</td>
</tr>
<tr>
<td>355</td>
<td>3</td>
</tr>
<tr>
<td>379</td>
<td>6</td>
</tr>
<tr>
<td>559</td>
<td>3</td>
</tr>
<tr>
<td>577</td>
<td>3</td>
</tr>
<tr>
<td>799</td>
<td>3</td>
</tr>
</tbody>
</table>

This is a total of 42 numbers.

So the answer to the question is 41 + 42 = 83.
Problem 10)

Suppose a circle of radius one is inscribed inside an equilateral triangle, at each vertex there is a second smaller circle tangent to the larger circle and two sides of the triangle, at each vertex there is a third smaller circle tangent to the second circle and two sides of the triangle, and so on with no end to this sequence of smaller and smaller circles at each vertex. What fraction of the triangle is occupied by the infinitely many circles (including the circle of radius 1)? That is, express the sum of the areas of the circles as a fraction of the area of the triangle.

[Problem submitted by Vin Lee, LACC Professor of Mathematics. Source: Vin Lee]

Solution: Orient the equilateral triangle with one of its sides horizontal and the opposite vertex above the horizontal side. First consider the circle of radius 1. Draw a line segment whose endpoints are the center of the circle and the lower right vertex of the triangle. Let \( x \) be the distance from the intersection of this line segment with the circle to the lower right vertex of the triangle. Now draw a vertical line segment from the center of the circle to the base of the triangle. Note that this forms a \( 30^\circ, 60^\circ, 90^\circ \) triangle whose hypotenuse has a length of \( x + 1 \) and shorter leg a length of 1, \( l \) be the length of the longer leg.

For this triangle \( \frac{vertical}{horizontal} = \frac{1}{l} = \frac{1}{\sqrt{3}} \rightarrow l = \sqrt{3} \)

\( \frac{hypotenuse}{vertical} = \frac{1 + x}{1} = \frac{2}{1} \rightarrow x = 1 \)
Now consider the first (largest) of the smaller circles approaching the lower right vertex. Draw a line segment from the center of this circle to the lower right vertex of the triangle. Let $r_1$ be the radius of this circle and $x_1$ be the distance from the intersection of the line segment with the circle to the lower right vertex of the triangle. Now draw a vertical line segment from the center of the circle to the base of the triangle. Note that this forms a $30^\circ, 60^\circ, 90^\circ$ triangle the length of whose hypotenuse is $r_1 + x_1$ and shorter leg $r_1$.

For this triangle \[ \frac{\text{hypotenuse}}{\text{vertical}} = \frac{r_1 + x_1}{r_1} = \frac{2}{1} \rightarrow r_1 = x_1 \]

Note that $2r_1 + x_1 = x = 1$. Therefore, $r_1 = x_1 = \frac{1}{3}$.

Next consider the second of the smaller circles approaching the lower right vertex. Draw a line segment from the center of this circle to the lower right vertex of the triangle. Let $r_2$ be the radius of this circle and $x_2$ be the distance from the intersection of the line segment with the circle to the lower right vertex of the triangle. Now draw a vertical line segment from the center of the circle to the base of the triangle. Note that this forms a $30^\circ, 60^\circ, 90^\circ$ triangle the length of whose hypotenuse is $r_2 + x_2$ and shorter leg $r_2$.

For this triangle \[ \frac{\text{hypotenuse}}{\text{vertical}} = \frac{r_2 + x_2}{r_2} = \frac{2}{1} \rightarrow r_2 = x_2 \]

Note that $2r_2 + x_2 = x_1 = \frac{1}{3}$. Therefore, $r_2 = x_2 = \frac{1}{9}$.

So the radii of the smaller circles at this vertex are $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$.

The sum if their areas is \[ \frac{\pi}{3^2} + \frac{\pi}{9^2} + \frac{\pi}{27^2} + \cdots = \sum_{n=0}^{\infty} \frac{\pi}{9^n} \left( \frac{1}{9} \right)^n = \frac{\pi}{\frac{9}{1} - \frac{1}{9}} = \frac{\pi}{8} \]

The area of all the circles within the equilateral triangle is $\pi + 3 \cdot \frac{\pi}{8} = 11 \cdot \frac{\pi}{8}$

Above we found that the height of the equilateral triangle is 3 and the base is $2\sqrt{3}$; so, its area is $3\sqrt{3}$.

The area of the circles divided by the area of the triangle is $\frac{11}{24\sqrt{3}} \pi$, which is approximately .8313247086, slightly less than $\frac{5}{6}$.