Test #1 **AMATYC Student Mathematics League** October/November 2009

1. Find the sum of the solutions to the equations $x^2 - 5x - 6 = 0$ and $x^2 + 4x + 3 = 0$ which DO NOT satisfy both equations at once.

A. -2 B. -1 C. 1 D. 2 E. 3

2. Four consecutive integers are substituted in every possible order for a, b, c, and d. Find the difference between the maximum and minimum values of ab + cd.

A. 1 B. 2 C. 3 D. 4 E. 5

3. The product of a number and b more than its reciprocal is y ($b > 0$). Express the number in terms of b and y.

A. $\frac{y-1}{b}$ B. $\frac{y+1}{b}$ C. $\frac{y}{b}-1$ D. $\frac{y}{b}+1$ E. $1-\frac{y}{b}$

4. If $f(x) = x^2 - x + 2$, find the sum of all x values satisfying $f(x-2) = 22$.

A. -5 B. -1 C. 1 D. 3 E. 5

5. Sue bikes 2.5 times as fast as Joe runs, and in 1 hr they cover a total of 42 miles. What is their combined distance if Sue bikes for 0.5 hr and Joe runs for 1.5 hr?

A. 27 B. 30 C. 33 D. 36 E. 39

6. The equation $a^4 + b^3 + c^2 = 2009$ (a, b, c positive integers) has a solution in which a and b are both perfect squares. Find $a + b + c$.

A. 20 B. 21 C. 32 D. 45 E. 50

7. How many 3-digit numbers have one digit equal to the average of the other 2?

A. 96 B. 100 C. 112 D. 120 E. 121

8. A rectangular solid has integer dimensions with length ≥ width ≥ height and volume 60. How many such distinct solids are there?

A. 6 B. 8 C. 10 D. 12 E. 15

9. $\frac{2 \sin x}{\cos x - \sin x \tan x} =$ A. tan 2x B. cot 2x C. tan x D. cot x E. sec x

10. If $x + \frac{1}{y} = 12$ and $y + \frac{1}{x} = \frac{3}{8}$, find the largest value of xy.

A. 1/4 B. 1/2 C. 1 D. 2 E. 4

11. In quadrilateral ABCD, P is a point in its interior such that $\angle DAP = \angle BAP$, $\angle CBP = \angle ABP$, and $\angle APB = 90°$. Quadrilateral ABCD must be which of the following?

A. trapezoid B. parallelogram C. rectangle D. B and C E. none of these

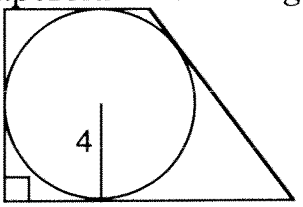
12. The sum of the squares of the three roots of $P(x) = 2x^3 - 6x^2 + 3x + 5$ is

A. 3 B. 6 C. 30 D. 33 E. 39

13. The value of $4^{\log_2\left(2^{1/4}2^{1/8}2^{1/16}\ldots\right)}$ is A. 1 B. $\sqrt{2}$ C. 2 D. $2\sqrt{2}$ E. 4

14. The figure shows a circle of radius 4 inscribed in a trapezoid whose longer base is three times the radius of the circle. Find the area of the trapezoid.

A. 72 B. 74 C. 76 D. 78 E. 80



15. In how many ways can six computers be networked so that each computer is directly connected to exactly two other computers, and all computers are connected directly or indirectly?

A. 24 B. 36 C. 48 D. 60 E. 120

16. The integer r > 1 is both the common ratio of an integer geometric sequence and the common difference of an integer arithmetic sequence. Summing corresponding terms of the sequences yields 7, 26, 90, ... The value of r is

A. 2 B. 4 C. 6 D. 8 E. 12

17. A hallway has 8 offices on one side and 5 offices on the other side. A worker randomly starts in one office and randomly goes to a second and then a third office (all three different). Find the probability that the worker crosses the hallway at least once.

A. $\frac{7}{13}$ B. $\frac{8}{13}$ C. $\frac{9}{13}$ D. $\frac{10}{13}$ E. $\frac{11}{13}$

18. Let S = {123, 124, ..., 987} be the set of all three-digit numbers with distinct nonzero digits. For which number N below does S contain at least two different numbers with the same three digits, both divisible by N?

A. 31 B. 37 C. 39 D. 41 E. 43

19. In square ABCD, AB = 10. The square is rotated 45° around point P, the intersection of $\overline{AC}$ and $\overline{BD}$. Find the area of the union of ABCD and the rotated square to the nearest square unit.

A. 117 B. 119 C. 121 D. 123 E. 125

20. The sum of the 100 consecutive perfect squares starting with a^2 ($a > 0$) equals the sum of the next 99 consecutive perfect squares. Find a. Write your answer in the corresponding blank on the answer sheet.

AMATYC Contest, Round 1, Oct.–Nov. 2009

① $(x-6)(x+1)=0 \Rightarrow 6, -1;\quad (x+3)(x+1)=0 \Rightarrow -3, -1;\quad 6+(-3)$ Ⓔ

② $x-1, x, x+1, x+2;\quad \max-\min = [(x+1)(x+2)+x(x-1)]-[(x-1)(x+2)+x(x+1)]$
$= (2x^2+2x+2)-(2x^2+2x-2) = 4$ Ⓓ

③ $x\left(\frac{1}{x}+b\right)=y$ Ⓐ
$1+xb=y,\quad x=\frac{y-1}{b}$

④ $f(x-2)=(x-2)^2-(x-2)+2=x^2-5x+8=22,$
$x^2-5x-14=0,\ x=7,-2;\ 7+(-2)=5$ Ⓔ

⑤ Joe $=x$, Sue $=\frac{5}{2}x$, $x+\frac{5}{2}x=42$, $x=12$, Sue$\left(\frac{1}{2}\right)$+Joe$\left(\frac{3}{2}\right)=33$ ~~Ⓓ~~ Ⓒ

⑥ Using TI-89, $\sqrt[3]{2009-4^4-x^2}=$ ⑨ or 12, $a+b+c=4+9+32$ Ⓓ

⑦

average	1	2	3	4	5	6	7	8	9	total
# of cases	5	11	17	23	25	19	13	7	1	121

Ⓔ

⑧

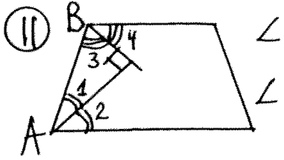
height	1	2	3	total
# of cases for w×l	5	3	2	10

Ⓒ

⑨ $\frac{2\sin x}{\cos x-\sin x\cdot\frac{\sin x}{\cos x}}$
$=\frac{2\sin x\cos x}{\cos 2x}=\tan 2x$ Ⓐ

⑩ $\begin{cases} xy+1=12y \\ xy+1=\frac{3}{8}x \end{cases} \Rightarrow x=32y,\ 32y+\frac{1}{y}=12,$
$(4y-1)(8y-1)=0,$ Ⓓ
$y=\{\frac{1}{4}, 8\},\ x=\{8, 4\},\ \max(xy)=2$

⑪

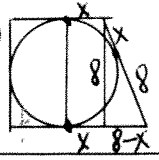


$\angle 1+\angle 3=90$
$\Downarrow$
$\angle A+\angle B=180$ Ⓐ

⑫ Vieth's Theorem
$\begin{cases} x+y+z=3 \\ xy+xz+yz=\frac{3}{2} \\ xyz=\frac{5}{2} \end{cases} \Rightarrow x^2+y^2+z^2+2\underbrace{(xy+xz+yz)}_{\frac{3}{2}}=9,$
$x^2+y^2+z^2=9-3$ Ⓑ

⑬ $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{2},$
$2^{\log_2(\sqrt{2})^2}=(\sqrt{2})^2=2$ Ⓒ

⑭



$8^2+(8-x)^2=(8+x)^2,\ x=2,$
$A=\frac{(a+b)h}{2}=\frac{12+(4+2)}{2}\cdot 8=72$ ~~Ⓒ~~ Ⓐ?

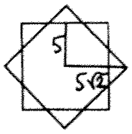
⑮ $C(5,2)\times 3\times 2=60$
Ⓓ

⑯ $a+b=7 \quad |\cdot(-r)$
$(a+r)+br=26 \quad |\cdot(1+r)$
$(a+2r)+br^2=90 \quad |\cdot(-1)$
$r+r^2-2r=-7r+26+26r-90$
$r^2-20r+64=0,\ r=$ ④, 16
$r=16 \Rightarrow$ ~~$b=\frac{1}{5}$~~, Ⓑ

⑰ $p=1-\frac{C(8,3)+C(5,3)}{C(13,3)}=1-\frac{66}{286}=\frac{10}{13}$ Ⓓ

⑱ 37 | 148, 481, 259, 592, 296, 629, 481, 814, … Ⓑ

⑲



$A=10^2+8\cdot\frac{(5\sqrt{2}-5)^2}{2}\approx 117$ Ⓐ

⑳ $a^2+(a+1)^2+(a+2)^2+\cdots+(a+99)^2=(a+100)^2+(a+101)^2+\cdots+(a+198)^2,$
$\sum_1^n k^2=\frac{n(n+1)(2n+1)}{6} \Rightarrow \sum_1^{a+99}k^2-\sum_1^{a-1}k^2=\sum_1^{a+198}k^2-\sum_1^{a+99}k^2 \ldots$
$\Rightarrow a^2-99\cdot 198a-99^2\cdot 199=0,\quad (a+99)(a-99\cdot 199)=0,$
~~$a=-99$~~ $(a>0)$, $a=99\cdot 199=\boxed{19,701}$